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PATH INDEPENDENT MULTIPLICATIVELY DECOMPOSABLE INEQUALITY MEASURES

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Foster and Shneyerov (2000) explore an additive decomposition property for inequality measures that they call path independent decomposition. They characterise the class of measures that have this property. Taking their results as a reference, we seek here to generalise the notion of decomposability, and to characterise inequality measures that admit path independent multiplicative decomposition. It is shown that all the measures in this family can be generated on the basis of Atkinson's index for value 1 of the inequality aversion parameter.

Keywords: Atkinson inequality measure, path independent decomposition.

(*JEL* C43, D31, D63)

1. Introduction

The purpose of this paper is the characterisation of the class of measures which, under certain basic hypotheses, are *path independent multiplicatively decomposable.*

A framework for studying the decomposability of an inequality measure must consider two things: firstly the definition of within- and between-group components and secondly the way in which those components aggregate to give rise to overall inequality.

J. Foster and A. Shneyerov (2000) take earlier studies (Shorrocks, 1980 and Anand, 1983) as a basis and call into question the traditional definition of within- and between-group terms. They notice that the components traditionally considered are not independent since variations in between-group inequality result in modifications not only in

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the between-group component but also in the within-group one, even though there may have been no change in within-group inequality. They explore an additive decomposition property for inequality measures that they call *path independent decomposition* and characterize the class of measures that have this property.

We believe that the first question concerning the proper definition of between- and within-group components is thus resolved[1]. The present study, taking into consideration these definitions, opens up the possibility of an alternative decomposition for equality measures as the product of the corresponding within- and between-group components.

In this context, firstly we characterise in Proposition 2 of Section 3 the single-parameter family of measures that, up to a positive power, admits a *path independent multiplicatively decomposition.* This family contains Atkinson's index for value 1 of the inequality aversion parameter (Atkinson, 1970). Moreover this index is the generator for the whole family. Section 2 summarises the hypotheses and the basic results that enable us to characterise this family.

Secondly taking Sala-i-Martin's (2002) article as a reference[2], the possibilities of this index are shown in regard to completing and detailing information in studies of inequality among populations and population subgroups (Section 4).

2. Preliminaries

We will consider a population consisting of $n \geq 2$ individuals. Individual i's income is denoted by $y_i \in \mathbb{R}_{++} = (0, \infty)$, $i = 1, ..., n$. An income distribution is represented by a vector $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_{++}$. We let $D = \cup_{n=1}^{\infty} \mathbb{R}^n_{++}$ represent the set of all finite dimensional income distribution and denote the mean and population size of any $y \in D$ by $\mu(y)$ and $n(y)$, respectively.

We will say that distribution $x \in D$ is a permutation of $y \in D$ if $x = \Pi y$ for some permutation matrix Π; that x is an m-replication of

[1] Shorrocks (1999) proposes an intermediate solution to define the different components using the Shapley value.

[2] We know that the world-wide database drawn up by Sala-i-Martín has come in for considerable criticism in regard to the source of his data and the non -tested hypotheses used to extrapolate missing data. We agree with most of this criticism, but we believe that the shortfalls of the database are lessened when the analysis is restricted to the G20 countries, where the data sources are acceptably reliable and there are practically no extrapolations.

y if $x = (y, y, ..., y)$ and $n(x) = mn(y)$ for some positive integer m, and that x is obtained from y by a simple increment if there exist i such that $x_i > y_i$ and $x_j = y_j$ for all $j \neq i$.

Finally, in the sequel q-order means will play a role. We define $m_q(y) = \left(1/_n \sum_{i=1}^{n} y_i^q\right)^{1/_q}$ for $q \in \mathbb{R}, q \neq 0$ and the geometric mean $m_0(y) = \left(\Pi_{i=1}^{n} y_i\right)^{1/_n}$. When $q = 1$ we obtain the average or arithmetic mean $\mu(y)$.

For the purposes of this paper, an inequality index is a real valued function $I : D \to \mathbb{R}$ which satisfies the following properties: 1)*Symmetry.* $I(x) = I(y)$ whenever x is a permutation of y; 2)*Normalisation.* $I(y) \geq 0$ for all $y \in D$; 3)*Replication Invariance.* $I(x) = I(y)$ whenever x is a replication of y; 4)*Scale Invariance.* $I(\lambda y) = I(y)$ for all $\lambda > 0$; 5)*Continuity.* For every $n > 0 : I(y)$ is continuos on $\mathbb{R}_{++}^n$. The equality index is taken as $E(y) = 1 - I(y)$.

A representative income function is a function $r : D \to \mathbb{R}$ satisfying the following restrictions: 1)*Symmetry.* $r(x) = r(y)$ whenever x is a permutation of y; 2)*Replication Invariance.* $r(x) = r(y)$ whenever x is a replication of y; 3)*Monotonicity.* $r(x) > r(y)$ whenever x is obtained from y by a simple increment; 4)*Linear homogeneity.* $r(x) = \alpha r(y)$ whenever $x = \alpha y$ for $\alpha > 0$; 5)*Normalisation.* $r(\mu, ..., \mu) = \mu$; 6) *Continuity.* For every $n > 0 : r(y)$ is continuos on $\mathbb{R}_{++}^n$.

Suppose that the population of n individuals is split into J mutually exclusive subgroups with income distribution $y^j = (y_1^j, ..., y_{nj}^j)$, mean incomes $\mu_j = \mu(y^j)$ and population sizes $n_j = n(y^j)$ for all $j = 1, ..., J$. Let inequality in group j be written $I_j = I(y^j)$. Let r be a representative income.

The *smoothed distribution* associated with $(y^1, ..., y^J)$ and r is defined by

$$y^B = \left(r\left(y^1\right)u_{n1}, \ldots, r\left(y^J\right)u_{nJ}\right)$$

where u_{nj} represents the unit vector $(1, 1, ..., 1)$ with n_j components. The smoothed distribution gives each person in a group its representative income.

The *standardised distribution* associated with $(y^1, ..., y^J)$ and r is defined by

$$y^W = r(y)\left(\frac{y^1}{r(y^1)}, \ldots, \frac{y^J}{r(y^J)}\right)$$

The standardised distribution re-scales each group distribution so that the representative incomes of all groups equal the overall representative income $r(y)$.

Either of these distributions may be used as a basis for decomposing total inequality into within and between-group terms. The first defines the between-group term, $I_B = I\left(y^B\right)$ and the second defines the within-group inequality, $I_W = I\left(y^W\right)$.

Foster and Shneyerov state that an inequality measure has a *path independent additive decomposition* if there is a representative income function r such that:

$$I\left(y^1, \ldots, y^J\right) = I\left(y^W\right) + I\left(y^B\right) \qquad [1]$$

for all $y^1, \ldots, y^j \in D$ and $J \geq 2$ and they conclude that this property characterises the following single parameter class, up to a scalar multiple:

$$I_q^*\left(y\right) = \begin{cases} \frac{1}{2} V_L\left(y\right) & if q{=}0 \\ \frac{1}{q} \ln\left(\frac{m_q(y)}{m_o(y)}\right) & if q{\neq}0 \end{cases} \qquad [2]$$

where $V_L\left(y\right) = \sum_{i=1}^n \left(\ln y^i - \ln m_o\left(y\right)\right)^2 / n$ is the variance of logarithms.

For this family of measures the decomposition given in [1] is expressed as follows:

$$I\left(y^1, \ldots, y^1\right) = I\left(y^w\right) + I\left(y^B\right) = \sum_{j=1}^{J} \frac{n_j}{n} I_j + I\left(y^B\right) \qquad [3]$$

3. Path independent multiplicatively decomposable inequality measures

The notion of multiplicative decomposability introduced in this paper is the following:

An inequality measure has a *path independent multiplicative decomposition* if there is a representative income function r such that

$$\left(1 - I\left(y^1, \ldots, y^J\right)\right) = \left(1 - I\left(y^W\right)\right)\left(1 - I\left(y^B\right)\right) \qquad [4]$$

or equivalently, in terms of equality indices:

$$E\left(y^1, \ldots, y^J\right) = E\left(y^W\right) E\left(y^B\right) \qquad [5]$$

PROPOSITION 1. *Let I be an inequality measure and let r be a representative income. Let $\bar{I} = 1 - e^{-I}$. The measure $\bar{I}$ has a path independent multiplicative decomposition for r if and only if I is a measure that has a path independent additive decomposition for r.*

PROOF. It is straightforward to show that $\bar{I}$ is an inequality measure. Assume that $\bar{I}$ admits a path independent multiplicative decomposition for r. From [4] and taking into account the definition of $\bar{I}$, we have:

$$e^{-I\left(y^1,\ldots,y^J\right)} = e^{-I\left(y^W\right)}e^{-I\left(y^B\right)} \quad [6]$$

Taking logarithms in equation [6] and operating, we have:

$$I\left(y^1,\ldots,y^J\right) = I\left(y^W\right) + I\left(y^B\right)$$

whence I is a measure that admits path independent additive decomposition for r.

Reciprocally, let I be an inequality measure that admits path independent additive decomposition for r. Considering $\bar{I} = 1 - e^{-I}$, operating and taking into account [1], we have:

$$1 - \bar{I}\left(y^1,\ldots,y^J\right) = e^{-I\left(y^1,\ldots,y^J\right)} = e^{-\left(I\left(y^W\right)+I\left(y^B\right)\right)} = \left(1 - \bar{I}\left(y^W\right)\right)\left(1 - \bar{I}\left(y^B\right)\right)$$

whence $\bar{I}$ admits path independent multiplicative decomposition for r.
■

PROPOSITION 2. *An inequality measure has a path independent multiplicative decomposition if and only if its equality index is a positive power of E_q with $q \in \mathbb{R}$, where:*

$$E_q\left(y\right) = \begin{cases} e^{-1/2 V_L(y)} \\ \left(\frac{m_o(y)}{m_q(y)}\right)^{1/q} \end{cases} \quad [7]$$

PROOF. Apply Proposition 1 to class [2] of path independent additively decomposable measures to obtain the single parameter family [7] expressed in terms of equality indices. ■

3.1 Some Remarks

Let us now restrict attention to the single-parameter family in [7].

1. For $q = 1, i.e.$ when we take the arithmetic mean as representative income, Atkinson's index for value 1 of the inequality aversion parameter, $E_1^A\left(y\right)$, is obtained. Furthermore, any measure from

[7] with $q \neq 0$ can be interpreted as a transformation of this index. Just consider a power distribution $y^q = (y^{1q}, ..., y^{nq})$, which transforms each income value into its $q - th$ power. For any $q \neq 0$, given that $m_q(y) = (m(y^q))^{1/q}$, it holds that

$$E_q(y) = \left[E_I^A(y^q)\right]^{1/q^2}$$

so E_q is a power of the value of index E_I^A for the $q - th$ power distribution.

And given that $\lim_{q \to 0} \left[E_1^A(y^q)\right]^{1/q^2} = e^{-1/2 V_L(y)} = E_0(y)$ the manner in which E_0 orders the distributions is approximately the manner in which E_I^A orders the $q - th$ power distributions for small $q\prime s$.

2. In practical studies it is convenient to have the decomposition of the overall measure into its components expressed analytically. For the measures in family [7] bearing in mind equations [3] and given that $E_q = 1 - I_q = e^{-I^*}$, the path independent multiplicative decomposition in [5] is therefore expressed as:

$$E_q(y) = E_q\left(y^W\right) E_q\left(y^B\right) = \Pi_{j=1}^J E_{qj}^{n_j/n} E_q\left(y^B\right) \qquad [8]$$

Equation [8] can be transformed into the following form:

$$\ln(E_q(y)) = \ln\left(E_q\left(y^W\right)\right) + \ln\left(E_q\left(y^B\right)\right) = \sum_{j=1}^J \frac{n_j}{n} \ln(E_{qj}) + \ln\left(E_q\left(y^B\right)\right) \qquad [9]$$

Taking the difference of equation [9] gives equation [10]:

$$\triangle \ln(E_q(y)) = \triangle \ln\left(E_q\left(y^W\right)\right) + \triangle \ln\left(E_q\left(y^B\right)\right) \qquad [10]$$

and since

$$\triangle \ln x = \ln x_t - \ln x_{t-1} \approx^{(x_t - x_{t-1})} /_{x_{t-1}} \qquad [11]$$

approximates the percentage change in x, equation [10] shows that the overall percentage rate of change in the equality can be expressed as the sum of the percentage changes in the respective within- and between- group equalities. The practical implication of the multiplicative decomposition is that these indices can be linearized so that the percentage change in these indices are additively decomposable.

3. One of the basic axioms considered for an inequality measure is the Dalton-Pigou Principle of Transfers. It can be proved that Eq satisfies the transfer principle if and only if $q \geq 1$.

4. On the other hand we must indicate that although normalisation was not explicitly required, all the indices in this family [7] are normalised between 0 and 1. Indeed if $q \geq 0$ then $m_q(y) \geq m_0(y)$, and if $q < 0$ then $m_q(y) \leq m_0(y)$, hence, in both cases $(m_q(y))^{1/q} \geq (m_0(y))^{1/q}$.

4. Application

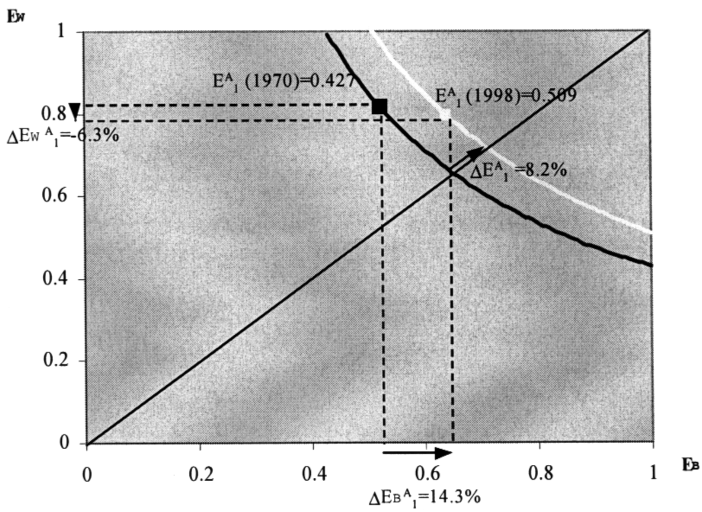
The paper by Sala-i-Martin (2002) estimates global income inequality for the G20 countries for every year between 1970-1998. Taking this paper as a reference this section seeks to show the possibilities of the Atkinson index for value 1 of the inequality aversion parameter in regard to completing and detailing information in studies of inequality among population subgroups. This index is not only decomposable (according to equation [8]) and has independent within- and between-group components but it is also normalised. Only with normalised measures variations in inequality can be worked out directly by calculating the variations in the relevant indices.

In this study the population subgroups are the countries, and therefore the between-country component assumes that all individuals in a country have the same per capita income and the within-country component takes into account within country differences in incomes.

Let us take the figures for this index, in terms of equality, for 1970: E_1^A=0.427, $E_{W_1^A}$=0.855, $E_{B_1^A}$=0.500, and for 1998: E_1^A=0.509, $E_{W_1^A}$=0.792, $E_{B_1^A}$=0.643. Taking into account that they vary within the range [0,1], these figures are a long way 1, which indicates the most egalitarian situation. It can also be observed that according to these indices global equality increased between 1970 and 1998 0.082, between-country equality component increased 0.143, and within-country equality decreased 0.063, and we can add that the extent of the increase in between-country equality is more than twice the decrease in within-country.

Moreover, it is possible to represent the multiplicative decomposition with both its components in graphic form. Consider a square with sides of unit length, as shown in Figure 1, and the level curves of function $E_1^A(y) = F\left(E_{B_1^A}, E_{W_1^A}\right) = E_{B_1^A} E_{W_1^A}$. On the x-axis we can represent the figures for $E_{B_1^A}$ and on the y-axis those for $E_{W_1^A}$.

FIGURE 1
Equality Decomposition for G20 countries between 1970-1998: Equality Atkinson Index (1)



Source: Data from Sala-i-Martin (2002)

Finally, the rate of increase of global equality is 0.192 in the period 1970-1998. As equations [10] and [11] show, this increase is due approximately to the 143.18 percent increase in the between-country equality and the 43.18 percent decrease in the within-country equality.

What we are seeking to do in demonstrating that Atkinson index E_1^A is *the only* multiplicatively decomposable path independet measure and exploring the properties satified by it, is to recover this index for practical work in this field, and we hope that this paper will convince the reader that this index is an important and useful tool for the study of inequality among populations and population subgroups.

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Resumen

Foster and Shneyerov (2000), exploran una propiedad de descomposición aditiva para las medidas de desigualdad que denominan descomposición independiente del camino, y caracterizan la clase de medidas que verifican esta propiedad. En este trabajo, tomando como referencia este resultado, se generaliza la noción de descomponibilidad, y se caracterizan las medidas de desigualdad que admiten una descomposición multiplicativa independiente del camino. Se prueba que todas las medidas de esta familia pueden ser generadas a partir del índice de Atkinson para el valor 1 del parámetro de aversión a la desigualdad.

Palabras clave: Medida de desigualdad de Atkinson, descomposición independiente del camino.

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